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Harmonic Analysis on Unitary Groups

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A theory of harmonic analysis on a metric group (G, d) is developed with the model of $U(\mathfrak{A})$, the unitary group of a C^*-algebra $\mathfrak{A}$, in mind. Essential in this development is the set $\hat{G}_d$ of contractive, irreducible representations of G, and its concomitant set $P_d(G)$ of positive-definite functions. It is shown that $\hat{G}_d$ is compact and closed in $\hat{G}$. The set $\hat{G}_d$ is determined in a number of cases, in particular when $G = U(\mathfrak{A})$ with $\mathfrak{A}$ abelian. If $\mathfrak{A}$ is an AW^*-algebra, it is shown that $\hat{G}_d$ is essentially the same as $\hat{\mathfrak{A}}$. Unitary groups are characterised in terms of a certain Lie algebra $\mathfrak{g}_u$, and several characterisations of $U(\mathfrak{A})$ when $\mathfrak{A}$ is abelian are given.

1. INTRODUCTION

The present paper is concerned with developing a harmonic analysis theory on metric groups, with particular emphasis on the unitary group $U(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$. Part of the motivation for this study is the evidence that some operator algebra properties can be interpreted in terms of such a unitary group. (For example, a result of de la Harpe [4] assrts that a von Neumann algebra $\mathfrak{A}$ is injective if and only if $U(\mathfrak{A})$ is "amenable" in the sense that there exists a left invariant mean on the space of right uniformly continuous functions on $U(\mathfrak{A})$.)

In developing such a theory, the following natural and interesting question arises: given the unitary group of a C^*-algebra, can we construct the C^*-algebra? A priori, it is not even clear that a C^*-algebra $\mathfrak{A}$ is uniquely determined by its unitary group $U(\mathfrak{A})$. (In fact, it is possible to have two non-isomorphic C^*-algebras with isometrically isomorphic unitary groups ((4.9)).) This question, and its concomitants, are investigated in the present paper. (Kadison's paper [14] on $U(\mathfrak{A})$, where $\mathfrak{A}$ is a factor, does not seem relevant in this context.)

The ideas developed in this investigation lead naturally to a version of harmonic analysis on metric groups with strong use of the metric compensating for the absence of Haar measure. Motivated by the special properties of the positive-definite function obtained by restricting to $U(\mathfrak{A})$ a state ϕ on a C^*-algebra $\mathfrak{A}$, we define in Section 3 a weak* compact convex set $P_d(G)$ of

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normalised positive-definite functions on a metric group (G, d). Associated with $P_d(G)$ is $\hat{G}_d$, the space of irreducible, contractive, unitary representations of G. We show (along with other results) that $\hat{G}_d$ is compact in the dual of G (as a discrete group), and determine $\hat{G}_d$ when G is $\mathbb{T}^n$ or $\mathbb{R}^n$ with a resonable metric.

The fourth section studies the relationship between $\hat{G}_d$ and $\hat{\mathfrak{A}}$, where $\mathfrak{A}$ is a C^*-algebra and $G = U(\mathfrak{A})$ with the metric induced by the norm of $\mathfrak{A}$. In essence, Theorem 4.4 asserts that $\hat{G}_d$ and $\hat{\mathfrak{A}}$ can be identified if $\mathfrak{A}$ is an AW^*-algebra. Theorem 4.2 determines $\hat{G}_d$ when $\mathfrak{A}$ is an abelian C^*-algebra, and implies that $\hat{G}_d$ is, in general, much bigger than $\hat{\mathfrak{A}}$. We also determine ((4.6)) which C^*-algebras have a given unitary group.

In the final section, we tackle the related problems: when is a metric group (G, d) a unitary group, and when is a connected metric group the identity component of a unitary group? To answer this, we introduce a certain "universal" C^*-algebra $\mathfrak{A}_d$ and associate with G a certain Lie algebra $\mathfrak{g}_u$, the *unitary Lie algebra* of G. When G is a connected Lie group, it is shown ((5.3)) that $\mathfrak{g}_u$ is a homomorphic image of the Lie algebra of G, and so is finite-dimensional. A solution to the second problem above in terms of $\mathfrak{A}_d$ and $\mathfrak{g}_u$ is given in (5.2). The paper concludes by examining the above two problems when G is abelian. A solution, involving Kronecker sets, to the first problem is given in (5.5); solutions to the second are given in (5.4) and (5.10). The first of these results involves the existence of a suitable "functional calculus" on the group. The second of these results relies on a curious result ((5.6)) on the triviality of a certain bundle associated with an abelian unitary group.

2. NOTATIONS

The symbols $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the sets of positive integers, integers, real numbers, and complex numbers, respectively. The closure of a set A in a topological space is denoted by A^-. If $\mathfrak{B}$ is a real or complex normed space, then $\mathfrak{B}_1$ and $\mathfrak{B}'$ are respectively, the closed unit ball and dual of $\mathfrak{B}$. If A is a subset of a vector space, then $\operatorname{Span} A$ and $\operatorname{co} A$ are respectively, the span and convex hull of A in the space; if A is convex, then $\operatorname{Ext} A$ is the set of extreme points of A. If X is a compact, Hausdorff space, then $C(X)$ and $M(X)$ are respectively the spaces of complex-valued continuous functions and regular Borel measures on X. We let $PM(X)$ be the set of probability measures in $M(X)$. The C^*-algebra of bounded linear operators on a (complex) Hilbert space $\mathfrak{H}$ is denoted by $B(\mathfrak{H})$.

All C^*-algebras discussed are assumed to have an identity element 1. The spectrum of an element x in a C^*-algebra $\mathfrak{A}$ is denoted by $\operatorname{Sp}(x)$. Further, $U(\mathfrak{A})$ and $\mathfrak{A}^{-1}$ are respectively the groups of unitary and invertible elements in $\mathfrak{A}$, and $S(\mathfrak{A})$ is the state space of $\mathfrak{A}$. The space of hermitian elements of $\mathfrak{A}$ is denoted by Her $\mathfrak{A}$, and $\mathrm{S}(\mathfrak{A}) = i$ Her $\mathfrak{A}$. The spectrum of $\mathfrak{A}$ is denoted by $\hat{\mathfrak{A}}$.

Throughout the paper, (G, d) *will stand for a metric group.* The identity of G is denoted by e, and G_e is the identity component of G. The set of unitary representations of G is denoted by Rep G, while the set of continuous unitary representations of G is denoted by $\mathrm{Rep}_c(G)$. (So if $\pi \in \mathrm{Rep}_c(G)$ is defined on a Hilbert space $\mathfrak{H}$, then the maps $x \to \pi(x)\,\xi$ are continuous for all $\xi \in \mathfrak{H}$.) We define $\mathrm{Rep}_n(G)$ to be the set of elements $\pi \in \mathrm{Rep}_c(G)$ for which the map $x \to \pi(x)$ is norm continuous. Of importance to us is $\mathrm{Rep}_d(G)$, the set of representations π of G for which

$$\|\pi(x) - \pi(y)\| \leqslant d(x, y)$$

for all $x, y \in G$. It is a trivial (though important) fact that $\mathrm{Rep}_d(G) \subset \mathrm{Rep}_n(G)$. If τ is a locally compact topology on G, then $\hat{G}_\tau$ is the set of equivalence classes of τ-continuous, irreducible representations in Rep G. When τ is discrete we write $\hat{G}$ for $\hat{G}_\tau$. It is easy to see that $\hat{G}_n$ and $\hat{G}_d$, the sets of equivalence classes of irreducible representations in $\mathrm{Rep}_n(G)$ and $\mathrm{Rep}_d(G)$, are well-defined subsets of $\hat{G}$. If $\pi \in \hat{G}$, then $\tilde{\pi} \in \hat{G}$ is the conjugate representation of G [11, p. 17; 6, p. 280]. If $A \subset \hat{G}$, then $A^{\sim} = \{\tilde{\pi} : \pi \in A\}$. Obviously, $\hat{G}_\tau$, $\hat{G}_n$, and $\hat{G}_d$ are invariant under the map $\pi \to \tilde{\pi}$.

The convex set of normalised positive-definite functions on G is denoted by $\Pi(G)$, and $P(G)$ is the set of continuous members of $\Pi(G)$. If τ is a locally compact topology on G, then $P_\tau(G)$ is the set of τ-continuous functions in $\Pi(G)$. If $\phi \in \Pi(G)$, then π_ϕ is the cyclic representation of $l_1(G)$ on a Hilbert space $\mathfrak{H}_\phi$ canonically associated with ϕ.

3. The Spaces $\hat{G}_d$ and $P_d(G)$

We define $P_d(G)$ to be the set of functions $\phi \in P(G)$ such that whenever $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $u_1, \ldots, u_n \in G$ and $u, v \in G$, we have

$$\sum \bar{\alpha}_j \alpha_i [2\phi(u_j^{-1} u_i) - \phi((uu_j)^{-1}(vu_i)) - \phi((vu_j)^{-1}(uu_i))] \leqslant d(u, v)^2 \sum \bar{\alpha}_j \alpha_i \phi(u_j^{-1} u_i). \tag{1}$$

Equivalent to (1) is the inequality:

$$(2 - d(u, v)^2) \sum \bar{\alpha}_j \alpha_i \phi(u_j^{-1} u_i) \leqslant 2 \,\mathrm{Re} \left(\sum \bar{\alpha}_j \alpha_i \phi((uu_j)^{-1}(vu_i)) \right).$$

(3.1) PROPOSITION. (i) *An element* $\phi \in \Pi(G)$ *is in* $P_d(G)$ *if and only if* $\pi_\phi \in \mathrm{Rep}_d(G)$.

(ii) *If* $\pi \in \mathrm{Rep}_d(G)$ *is defined on* $\mathfrak{H}$ *and* $\xi \in \mathfrak{H}$ *has norm* 1, *then the function* $u \to (\pi(u)\,\xi, \xi)(u \in G)$ *is in* $P_d(G)$.

(iii) *If* $\phi \in P_d(G)$, *then* $|\phi(u) - \phi(v)| \leqslant d(u, v)$ *for all* $u, v \in G$. *Further, if G is abelian and* $\phi \in \hat{G}$, *then* $\phi \in \hat{G}_d$ *if and only if* $|\phi(u) - \phi(v)| \leqslant d(u, v)$ *for all* $u, v \in G$.

(iv) $P_d(G)$ *is an equicontinuous convex set of functions on* (G, d). *If* τ *is a locally compact group topology on G which is stronger than the metric topology and if* $L_1(G)$ *and* $L_\infty(G)$ *are defined with respect to* τ, *then the relative* $\sigma(L_\infty(G), L_1(G))$ *topology on* $P_d(G)$ *coincides with the pointwise topology, and* $P_d(G)$ *is compact for this topology.*

Proof. (i) and (ii). Suppose that $\phi \in P_d(G)$ and let $Q\colon l_1(G) \to \mathfrak{H}_\phi$ be the canonical linear mapping. If $\alpha_1,..., \alpha_n \in \mathbb{C}$, $u_1,..., u_n \in G$, $u, v \in G$ and $\xi = Q(\Sigma\, \alpha_i u_i)$, then $\|(\pi_\phi(u) - \pi_\phi(v))\,\xi\|^2$ and $d(u, v)^2\, \|\xi\|^2$ are respectively the left- and right-hand sides of (1). Conversely, if $\pi_\phi \in \mathrm{Rep}_d(G)$ and $\eta \in \mathfrak{H}_\phi$ is such that $\phi(u) = (\pi_\phi(u)\, \eta, \eta)$, then, unravelling the inequality

$$\left\|(\pi_\phi(u) - \pi_\phi(v))\left(\sum \alpha_i \pi_\phi(u_i)\right)(\eta)\right\|^2 \leqslant \|\pi_\phi(u) - \pi_\phi(v)\|^2 \left\|\sum \alpha_i \pi_\phi(u_i)(\eta)\right\|^2$$

and using the fact that $\pi_\phi \in \mathrm{Rep}_d(G)$, we obtain (i). So $\phi \in P_d(G)$. This also proves (ii).

(iii) In the above notation, if $\phi \in P_d(G)$ we have, using (i),

$$|\phi(u) - \phi(v)| = |((\pi_\phi(u) - \pi_\phi(v))\, Qe, Qe)| \leqslant \|\pi_\phi(u) - \pi_\phi(v)\| \leqslant d(u, v).$$

Now suppose that G is abelian and that $\phi \in \hat{G}$ satisfies the inequality: $|\phi(u) - \phi(v)| \leqslant d(u, v)$ for all $u, v \in G$. Then π_ϕ can be identified with ϕ, and $\phi \in P_d(G)$ by (i).

(iv) The convexity of $P_d(G)$ follows immediately from (1), and the equicontinuity of $P_d(G)$ follows from (iii). Now let τ be as in the statement of (iv). The weak* topology $\sigma(L_\infty(G), L_1(G))$ on $P_d(G)$ coincides with the topology of compact convergence on (G, τ) [6, p. 291]. The equicontinuity of $P_d(G)$ and the Arzèla–Ascoli theorem imply that if S is τ-compact in G, then $\{\phi_{|S} : \phi \in P_d(G)\}$ is norm relatively compact in $C(S)$. It follows that $P_d(G)$ is relatively compact in the space of τ-continuous, complex-valued functions on G with the compact convergence topology. Now if $\phi_\delta \to \phi$ in the latter topology, then ϕ satisfies (1) if every ϕ_δ does. So $P_d(G)$ is compact as required. Finally, the pointwise topology on $P_d(G)$ is weaker than the compact convergence topology, and so they have to coincide. ∎

(3.2) *Notes.* Since (G, d) may not be locally compact, the only reasonable topology for $P_d(G)$ is the relative $\sigma(l_\infty(G), l_1(G))$ topology, i.e., the pointwise topology. Result (iv) is encouraging since if the metric topology does happen to be locally compact, then the pointwise topology on $P_d(G)$ is "correct" from the point of view of harmonic analysis on locally compact groups.

The second assertion of (iii) is false if $\hat{G}$ and $\hat{G}_d$ are respectively replaced by $\Pi(G)$ and $P_d(G)$. For example, let $G = \mathbb{T}$, the circle group, d be the usual "modulus" metric, and $\phi: \mathbb{T} \to \mathbb{C}$ be given by $\phi(z) = (1 - \sum_{n=1}^{\infty} 3^{-n}) + \sum_{n=1}^{\infty} 3^{-n} z^n$.

By Bochner's theorem, $\phi \in \Pi(G)$ and one readily checks that for $z, t \in \mid T$,

$$|\phi(z) - \phi(t)| \leqslant |z - t| \sum_{1}^{\infty} n 3^{-n} \leqslant |z - t|.$$

However, it follows from (3.7) and (3.10)(ii) that $\phi \notin P_d(G)$.

In our next result, $\hat{G}_\tau$ is given the usual Fell topology.

(3.3) Proposition. (i) $\operatorname{Ext} P_d(G) \subset \operatorname{Ext} \Pi(G)$.

(ii) *If τ is as in* (3.1)(iv), *then $\hat{G}_d$ is a closed compact subset of $\hat{G}_\tau$, and the relative topology of $\hat{G}_d$ in $\hat{G}_\tau$ is independent of τ.*

Proof. (i) Let $\phi \in \operatorname{Ext} P_d(G)$, $\eta \in \mathfrak{H}_\phi \sim \{0\}$ and $\mathfrak{K} = (\pi_\phi(l_1(G))\, \eta)^-$. Suppose that $\mathfrak{K} \neq \mathfrak{H}_\phi$. Let $\xi \in \mathfrak{H}_\phi$ be such that $\phi(u) = (\pi_\phi(u)\, \xi, \xi)$ $(u \in G)$ and write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \mathfrak{K}$ and $\xi_2 \in \mathfrak{K}^\perp$. Let $\phi_i(u) = (\pi_\phi(u)\, \xi_i, \xi_i)$. Then $\phi_i / \|\xi_i\|^2 \in P_d(G)$ by (3.1)(ii) and ϕ is the convex combination $\sum \|\xi_i\|^2 \phi_i$. A contradiction results, and π_ϕ is irreducible. Hence $\phi \in \operatorname{Ext} \Pi(G)$ [6, 2.5.4].

(ii) We first show that $\hat{G}_d$ is closed in $\hat{G}_\tau$. Let π belong to the closure of $\hat{G}_d$ in $\hat{G}_\tau$ and let ϕ be a positive-definite function associated with π with $\phi(e) = 1$. By [6, 18.1.5], we can find $\{\phi_\delta\}$ in $P_d(G)$ with $\phi_\delta \to \phi$ uniformly on τ-compact sets. Then $\phi \in P_d(G) \cap \operatorname{Ext} P_\tau(G)$ and since $P_d(G) \subset P_\tau(G)$, we have $\phi \in \operatorname{Ext} P_d(G)$. Then π and π_ϕ are in $\hat{G}_\tau$, and ϕ is associated with both. So π and π_ϕ are equivalent and so, with the usual abuse of notation, $\pi \in \hat{G}_d$. So $\hat{G}_d$ is closed in $\hat{G}_\tau$. Using [6, 3.4.12] and (3.1)(iv), the relative topologies induced on $\hat{G}_d$ as subsets of $\hat{G}_\tau$ and $\hat{G}$ coincide. By the earlier part of the proof $\hat{G}_d$ is a closed subset of the component space $\hat{G}$, and so $\hat{G}_d$ is compact in $\hat{G}_\tau$. ∎

(3.4) COROLLARY. *If (G, d) is compact, then $\hat{G}_d$ is finite.*

Proof. The dual of G is discrete. ∎

(3.5) *Note.* If F is a finite subset of $\hat{G}_n$, then there exists a metric δ on G, giving the same topology as d, such that $F \subset \hat{G}_\delta$. Indeed, we can define

$$\delta(x, y) = \max\{d(x, y), \max\{\|\pi(x) - \pi(y)\|: \pi \in F\}\}.$$

(3.6) PROPOSITION. *Let (G, d) be locally compact and $\pi \in \hat{G}_d$. If either G is connected or G contains a large compact subgroup, then π is finite-dimensional.*

Proof. If G is connected, then $\pi \in \mathrm{Rep}_n(G)$, and so is finite-dimensional [22, 9].

Now suppose that G contains a large compact subgroup K and let $d' = d_{|K \times K}$. We can write the Hilbert space $\mathfrak{H}$ of π as $\oplus_\alpha \mathfrak{H}_\alpha$, where each $\mathfrak{H}_\alpha$ is finite-dimensional and the representation π_α, where $\pi_\alpha(k) = \pi(k)_{|\mathfrak{H}_\alpha}$ $(k \in K)$ is irreducible. Obviously, each $\pi_\alpha \in \hat{K}_{d'}$. Since K is large, we have that for each $\Phi \in \hat{K}_d$, the set $\{\alpha: \pi_\alpha = \Phi\}$ is finite. Since $\hat{K}_{d'}$ is finite ((3.4)), it follows that $\mathfrak{H}$ is finite-dimensional. ∎

Examples of locally compact groups with large compact subgroups are given in [23, 4.5.2]. Our next result shows that if G is abelian, then the study of $P_d(G)$ reduces to that of $\hat{G}_d$.

(3.7) PROPOSITION. *Let G be abelian. Then $P_d(G)$ is canonically identified with $PM(\hat{G}_d)$.*

Proof. By (3.3) $\mathrm{Ext}\, P_d(G) = \hat{G}_d$, and $\hat{G}_d$ is compact. It follows from [19, Propositions 1.1 and 1.2] that $P_d(G)$ can be identified with $PM(\hat{G}_d)$. ∎

We now discuss $\hat{G}_d$ when G is a connected, abelian, Lie group. The group G is, of course, of the form $\mathbb{R}^n \times \mathbb{T}^m$, but it is convenient to formulate the argument in abstract form. We shall suppose for the remainder of this section that d is invariant, and that for all X in the Lie algebra $\mathfrak{g}$ of G, we have

$$\limsup_{t \to 0+} (d(\exp(tX), e)/t) < \infty.$$

Now the function p_X, where $p_X(t) = d(\exp tX, e)$ for $t > 0$, is subadditive and hence by [12, Theorem 7.11.1], $\lim_{t \to 0+} (p_X(t)/t)$ exists, and equals $\sup_{t>0} (p_X(t)/t)$. Let $\|X\| = \lim_{t \to 0+} (p_X(t)/t)$. If $\|X\| = 0$, then $p_X(t) = 0$, so that $\exp tX = e$ for all t, and $X = 0$. It is left as a routine exercise to check that $\|\cdot\|$ is a norm on the real vector space $\mathfrak{g}$.

Let $W' = \{F \in (\mathfrak{g}')_1 : F(\ker \exp) \subset 2\mathbb{Z}\pi\}$.

(3.8) PROPOSITION. *There is a homeomorphism T from $\hat{G}_d$ onto a subset of W; the mapping T is given by:*

$$\exp[itT(\phi)(X)] = \phi(\exp tX) \qquad (\phi \in \hat{G}_d, X \in \mathfrak{g}, t \in \mathbb{R}). \tag{2}$$

Proof. Note that $T(\phi)$ is essentially just the differential representation of ϕ. It is routine to check, using the fact that G is abelian, that $T(\phi) \in \mathfrak{g}$. Since exp is surjective, T is one-to-one, and T^{-1} is continuous. If $\phi_\delta \to \phi$ in $\hat{G}_d$, then $\phi_\delta \to \phi$ uniformly on compacta (3.1)(iv); so if $X \in \mathfrak{g}$ and τ is the usual topology on $\mathbb{R}$, then $T(\phi_\delta)(X)\hat{} \to T(\phi)(X)\hat{}$ in $\mathbb{R}_\tau\hat{} = \mathbb{R}$, and so $T(\phi_\delta) \to T(\phi)$ in $\mathfrak{g}'$. So T is a homeomorphism. It is immediate from (2) that $T(\phi)(\ker \exp) \subset 2\mathbb{Z}\pi$. That $T(\phi) \in (\mathfrak{g}')_1$ and so to W follows from:

$$|T(\phi)(X)| = \lim_{t \to 0+} (|\phi(\exp tX) - 1|/t) \leqslant \lim_{t \to 0+} d(\exp tX, e)/t = \|X\|,$$

where we have used (3.1)(iii). ∎

(3.9) PROPOSITION. *Suppose that for each $X \in \mathfrak{g}$, there exists $\delta_X > 0$ such that*

$$d(\exp tX, e) = |t|\, d(\exp X, e) \qquad (|t| < \delta_X). \tag{3}$$

Then $T(\hat{G}_d) = W$.

Proof. Let $F \in W$. Since $F(\ker \exp) \subset 2\mathbb{Z}\pi$, we can define $\phi \in \hat{G}$ by: $\phi(\exp X) = \exp(iF(X))$. Now the function $t \to |\exp(itF(X)) - 1|$ is subadditive; using [12, Theorem 7.11.1] and (3), we have

$$\begin{aligned} |\phi(\exp X) - 1| = |\exp(iF(X)) - 1| &\leqslant \sup_{t > 0} (|\exp(itF(X)) - 1|/t) \\ &= \lim_{t \to 0+} (|\exp(itF(X)) - 1|/t) = |F(X)| \leqslant \|X\| \\ &= \lim_{t \to 0+} (d(\exp tX, e)/t) = d(\exp X, e). \end{aligned}$$

So $\phi \in \hat{G}_d$ by (3.1)(iii), and the desired result follows since $T(\phi) = F$. ∎

We now apply (3.8) and (3.9) to determine $P_d(G)$ when G is $\mathbb{R}^m$ or $\mathbb{T}^n$ with a suitable metric.

(3.10) PROPOSITION. (i) *If $1 \leqslant p \leqslant \infty$, $G = \mathbb{R}^m$ and d is the usual d_p-metric, then $P_d(G)$ can be identified with $PM(U_q)$, where $p^{-1} + q^{-1} = 1$, and U_q is the closed unit ball of the Banach space $(\mathbb{R}^m, \|\cdot\|_q)$.*

(ii) *If $1 \leqslant p \leqslant \infty$, $G = \mathbb{T}^n$ and d is the d_p-metric, then $P_d(G)$ can be* *identified with* $PM(K \cup \{0\} \cup -K)$, *where* K *is the discrete set* $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$ *in* $\mathbb{R}^n$.

Proof. (i) In this case, $\mathfrak{g} = \mathbb{R}^m$ and exp is the identity map. The equality (3) is satisfied, and the norm induced by d on G is just $\|\cdot\|_p$. Further, W is just the closed unit ball U_q of $\mathfrak{g}'$. The desired result follows from (3.7) and (3.9).

(ii) Here $\mathfrak{g} = \mathbb{R}^n$ and $\exp((a_1, \ldots, a_n)) = (\exp(ia_1), \ldots, \exp(ia_n))$. Further

$$\|(a_1, \ldots, a_n)\| = \lim_{t \to 0+} t^{-1} \left(\sum (|e^{ita_j} - 1|)^p\right)^{1/p} = \|(a_1, \ldots, a_n)\|_p.$$

Now $\ker \exp = (2\mathbb{Z}\pi)^n$, and so by (3.8), $T(\hat{G}_d) \subset \{\boldsymbol{b} \in \mathbb{R}^n : \|\boldsymbol{b}\|_q \leqslant 1, b_i \in \mathbb{Z}\} = K \cup \{0\} \cup -K$. Recalling that $|z^{-1} - t^{-1}| = |z - t|$ for all $z, t \in \mathbb{T}$, one easily checks that $T(\hat{G}_d) = K \cup \{0\} \cup -K$, and the desired result follows. ∎

(3.11) *Note.* In general, $T(\hat{G}_d)$ does not equal W. For example, if we take $G = \mathbb{T}$ and define $d(z, t)$ to be $|z - t|$ if $|z - t| \leqslant \frac{3}{2}$, and to be $\frac{3}{2}$ if $|z - t| > \frac{3}{2}$, then the norm on $\mathfrak{g} = \mathbb{R}$ is just the modulus, so that $W = \{-1, 0, 1\}$. Yet $\hat{G}_d = \{1\}$, so that $T(\hat{G}_d) = \{0\}$.

4. REPRESENTATIONS OF THE UNITARY GROUP OF A C^*-ALGEBRA

Throughout this section, $\mathfrak{A}$ is a C^*-algebra and $G = U(\mathfrak{A})$. The metric d on G is that inherited from the norm of $\mathfrak{A}$. If H is a subgroup of G, then, abusing notation, we shall use "d" for the restriction of d to $H \times H$. Since $\mathfrak{A} = \operatorname{Span} G$, we have that $\hat{G}_{\mathfrak{A}} = \{\pi|_G : \pi \in \hat{\mathfrak{A}}\}$ is a subset of $\hat{G}_d$. The subset $\{\phi|_G : \phi \in S(\mathfrak{A})\}$ of $P_d(G)$ is denoted by $P_{\mathfrak{A}}(G)$.

The following result is a routine consequence of the theory of Banach Lie algebras (e.g., [1, III, 3.10, Corollary 2]) and its proof is omitted. Recall that G_e is generated by $\{\exp(x) : x \in \operatorname{Sk} \mathfrak{A}\}$.

(4.1) PROPOSITION. (i) *The Lie algebra of G is* Sk $\mathfrak{A}$.

(ii) *If* $\mathfrak{B}$ *is a* C^**-algebra and* $Q: G_e \to H$ *is a homomorphism such that* $d(Q(u), Q(v)) \leqslant d(u, v)$ *for all* $u, v \in G_e$, *then there exists a Lie homomorphism* $\beta: \operatorname{Sk} \mathfrak{A} \to \operatorname{Sk} \mathfrak{B}$ *with* $\|\beta\| \leqslant 1$ *and such that* $\exp(\beta(x)) = Q(\exp x)$ $(x \in \operatorname{Sk} \mathfrak{A})$.

The smallest possible that $\hat{G}_d$ could be is $\hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}$. We see from (3.10)(i) that this is the case when $\mathfrak{A} = \mathbb{C}^n$, and we shall show ((4.4)) that it is true when $\mathfrak{A}$ is an AW^*-algebra. The next result entails that it is not true in general when $\mathfrak{A}$ is abelian.

(4.2) THEOREM. *Let X be compact Hausdorff, $\mathscr{S}$ be the algebra of open and closed subsets of X and $\mathfrak{A} = C(X)$. Let*

$$S_{\mathscr{S}}(\mathfrak{A}) = \{\mu \in S(\mathfrak{A}) : \mu(\mathscr{S}) = \{0, 1\}\}.$$

Then $\hat{G}_d$ is canonically identified with the compact space

$$H^1(X, \mathbb{Z})^{\wedge} \times (S_{\mathscr{S}}(\mathfrak{A}) \cup \{0\} \cup -S_{\mathscr{S}}(\mathfrak{A})),$$

where $H^1(X, \mathbb{Z})$ is the first Cech cohomology group of X.

Proof. Let $\gamma \in (G_e)_d^{\wedge}$ and (using (4.1)(ii)) let $\alpha \in (\mathrm{Her}\,\mathfrak{A})'$ be such that $\|\alpha\| \leqslant 1$ and $\gamma(\exp(ih)) = \exp(i\alpha(h))$. If $E \in \mathscr{S}$, then $\exp(i2\pi\alpha(E)) = \gamma(\exp(2\pi i X_E)) = 1$, so that $\alpha(E) \in \mathbb{Z}$. Since $\|\alpha\| \leqslant 1$, $\alpha(E) \in \{-1, 0, 1\}$. Indeed we either have $\alpha(\mathscr{S}) = \{0, 1\}$ or $\alpha(\mathscr{S}) = \{0, -1\}$ or $\alpha(\mathscr{S}) = \{0\}$. It follows that $\alpha \in S_{\mathscr{S}}(\mathfrak{A}) \cup \{0\} \cup -S_{\mathscr{S}}(\mathfrak{A})$.

Now suppose that $\beta \in S_{\mathscr{S}}(\mathfrak{A}) \cup \{0\} \cup -S_{\mathscr{S}}(\mathfrak{A})$. If $h, k \in \mathrm{Her}\,\mathfrak{A}$ and $\exp(ih) = \exp(ik)$, then $(h - k)$ is of the form $2\pi \sum n_i \chi_{E_i}$, where $n_i \in \mathbb{Z}$ and $E_i \in \mathscr{S}$. So $\exp(i\beta(h)) = \exp(i\beta(k))$. We can therefore define $\delta \in G_e^{\wedge}$ by setting $\delta(\exp(ih)) = \exp(i\beta(h))$. We now show that $\delta \in (G_e)_d^{\wedge}$. It is sufficient to show that

$$|\delta(\exp(ih) - 1| \leqslant \|\exp(ih) - 1\|. \tag{4}$$

Suppose first that $\beta \in S_{\mathscr{S}}(\mathfrak{A})$. If $\|\exp(ih) - 1\| = 2$, then (4) is trivially true. Suppose then that $\|\exp(ih) - 1\| = k < 2$. Let $E = \{z \in \mathbb{T} : |z - 1| \leqslant k\}$ and $F = \{\lambda \in \mathbb{R} : \exp(i\lambda) \in E\}$. Then F is the disjoint union of closed intervals F_n, where each F_n is open and closed in F and $E = \{\exp(i\lambda) : \lambda \in F_n\}$. Then X is the disjoint union of the sets $E_n = h^{-1}(F_n) \in \mathscr{S}$, and since $\beta \in S_{\mathscr{S}}(\mathfrak{A})$, there exists exactly one N with $\beta(E_N) = 1$. Since E_N supports β and $a\chi_{E_N} \leqslant h|_{E_n} \leqslant b\chi_{E_n}$, where $F_N = [a, b]$, we have $\beta(h) \in F_N$. Hence (4) holds. If $\beta \in -S_{\mathscr{S}}(\mathfrak{A})$, then $-\beta \in S_{\mathscr{S}}(\mathfrak{A})$, and applying the above result, the equality (4) holds. If $\beta = 0$, then $\delta = 1$ and (4) is trivial.

So we can identify $(G_e)_d^{\wedge}$ with $S_{\mathscr{S}}(\mathfrak{A}) \cup \{0\} \cup -S_{\mathscr{S}}(\mathfrak{A})$.

Now every element of G_e is of the form $\exp(ih)$, so that G_e is a divisible subgroup of G. Hence [10, (A.8)] G is a direct product $G_e \times K$. Now K is isomorphic with G/G_e and the map $(f + \exp \mathfrak{A}) \to (f/|f| + G_e)$ is an isomorphism from $\mathfrak{A}^{-1}/\exp \mathfrak{A} = H^1(X, \mathbb{Z})$ onto G/G_e. So K is isomorphic with $H^1(X, \mathbb{Z})$. It remains to show that $\hat{G}_d$ is isomorphic with $(G_e)_d^{\wedge} \times \hat{K}$. To this end, note that every element of $\hat{G}_d$ gives rise to an element of $(G_e)_d^{\wedge} \times \hat{K}$ in the obvious way. Conversely, let $(\gamma, \delta) \in (G_e)_d^{\wedge} \times \hat{K}$ and identify (γ, δ) with an element $\sigma \in \hat{G}$. Let $u \in G$. If $u \in G_e$, then

$$|\sigma(u) - 1| = |\gamma(u) - 1| \leqslant d(u, e).$$

If $u \notin \mathcal{G}_e$, then $\mathrm{Sp}(u) = \mathbb{T}$, and $|\sigma(u) - 1| \leqslant 2 = d(u, e)$. So $\sigma \in \hat{G}_d$, and the proof is complete. ∎

(4.3) *Note.* It follows from (4.2) that if X is connected and α is a state on $C(X)$, then there exists $\gamma \in \hat{G}_d$ such that $\gamma(\exp(ih)) = \exp(i\alpha(h))$ for all $h \in \mathrm{Her}\, C(X)$. The elements in $\hat{G}_{\mathfrak{A}}$ coming in this way from a state α are very "thin": indeed the state α has to be a point mass. So in general, $\hat{G}_{\mathfrak{A}}$ is a small subset of $\hat{G}_d$. However, when X is totally disconnected (i.e., $\mathscr{S}$ is a base for the topology on X), then

$$\hat{G}_d = \hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}.$$

(4.4) THEOREM. *Let $\mathfrak{A}$ be an AW^*-algebra (not necessarily abelian). Then $\hat{G}_d = \hat{G}_{\mathfrak{A}} \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}$.*

Proof. We have to show that $\hat{G}_d \subset (\hat{G}_{\mathfrak{A}}) \cup \{1\} \cup (\hat{G}_{\mathfrak{A}})^{\sim}$. Let $\pi \in \hat{G}_d$ and identify $\mathbb{T}1 \subset G$ with $\mathbb{T}$. Since π is irreducible on its Hilbert space $\mathfrak{H}$, we have $\pi(T) \subset \mathbb{C}I$, where I is the identity of $B(\mathfrak{H})$. So there exists $n \in \mathbb{Z}$ with $\pi(\lambda) = \lambda^n (\lambda \in \mathbb{T})$. Since $\pi|_{\mathbb{T}} = \hat{\mathbb{T}}_\rho$, where ρ is the modulus metric on $\mathbb{T}$, it follows from (3.10)(ii) that $n \in \{-1, 0, 1\}$. We deal with the cases $n = 1$, $n = 0$ and $n = -1$ separately. Let β: $\mathrm{Sk}\, \mathfrak{A} \to \mathrm{Sk}(B(\mathfrak{H}))$ be the map of (4.1)(ii) with $Q = \pi$.

Suppose that $n = 1$. Then

$$\exp(t\beta(i1)) = \pi(\exp(it1)) = \pi(\exp(it)\ 1) = \exp(it)\, I$$

so that $\beta(i1) = iI$. Let $\mathfrak{B}$ be an abelian, AW^*-subalgebra of $\mathfrak{A}$ containing 1, let $H = U(\mathfrak{B})$, and $\pi_H = \pi|_H$. Since β is a Lie homomorphism, it follows that $\beta(\mathrm{Sk}\, \mathfrak{B})$ is an abelian subset of $\mathrm{Sk}\, B(\mathfrak{H})$ and so is contained in an abelian, unital, C^*-subalgebra $\mathfrak{C}$ of $B(\mathfrak{H})$. Let $\gamma \in \hat{\mathfrak{C}}$. Clearly $\gamma \circ \pi_H \in \hat{H}_d$ and $\gamma \circ \pi_H(\exp(ih)) = \gamma[\exp(\beta(ih))] = \exp[\gamma \circ \beta(ih)]$. Let $\alpha_\gamma \in (\mathrm{Her}\, \mathfrak{B})'$ be given by: $i\alpha_\gamma(h) = \gamma \circ \beta(ih)$. Then from (4.2) and the fact that $\beta(i1) = iI$, we have, using an obvious notation, that $\alpha_\gamma \in S_{\mathscr{S}}(\mathfrak{B})$. Now $\mathfrak{B} = C(X)$, where X is Stonian. So $\mathscr{S}$, the family of open and closed subsets of X, is a base for the topology of X. Using the regularity of the measure α and the fact that $\alpha_\gamma(\mathscr{S}) = \{0, 1\}$, we see that α_γ is a point mass. Let $\alpha \in (\mathrm{Her}\, \mathfrak{A})'$ be given by $i\alpha(h) = \beta(ih)$. If $h \in \mathrm{Her}\, \mathfrak{B}$, then, since α_γ is multiplicative, $\gamma(\alpha(h^2)) = \gamma(\alpha(h)^2)$, and $\alpha(h)^2 = \alpha(h)^2$. Since every $h \in \mathrm{Her}\, \mathfrak{A}$ is an element of such an algebra $\mathfrak{B}$, we have $\alpha(h^2) = \alpha(h)^2$ for all $h \in \mathrm{Her}\, \mathfrak{A}$. Now define Φ: $\mathfrak{A} \to B(\mathfrak{H})$ by: $\Phi(h + ik) = \alpha(h) + i\alpha(k)(h, k \in \mathrm{Her}\, \mathfrak{A})$. Routine algebra shows that Φ is a *-homomorphism. Further, with γ and h as above,

$$\gamma \circ \pi(\exp(ih)) = \exp(i\alpha_\gamma(h)) = \alpha_\gamma(\cos h) + i\alpha_\gamma(\sin h) = \gamma \circ \Phi(\exp(ih)).$$

So π and Φ coincide on G_e. Since $\mathfrak{A}$ is an AW^*-algebra, we have $G_e = G$. Since $\mathrm{Span}\, G = \mathfrak{A}$, it follows that $\Phi \in \hat{\mathfrak{A}}$, and hence $\pi \in \hat{G}_{\mathfrak{A}}$.

If $n=-1$, then the conclusion of the above argument applies to $\tilde{\pi}$, so that $\pi \in (\hat{G}_{\mathfrak{A}})^{\sim}$.

Finally, suppose that $n=0$. Then $\alpha(1)=0$. Let $\mathfrak{B}$, X, γ and $\mathscr{S}$ be as earlier. Since $\alpha_\gamma \in S_{\mathscr{S}}(\mathfrak{B}) \cup \{0\} \cup -S_{\mathscr{S}}(\mathfrak{B})$ and $\alpha_\gamma(1)=0$, we must have $\alpha_\gamma(E)=0$ for all $E \in \mathscr{S}$, and so $\alpha_\gamma=0$. It follows that $\alpha(h)=0$ for all $h \in \operatorname{Her} \mathfrak{A}$, so that $\pi(\exp(ih))=I$. Hence π is the trivial one-dimensional representation of G, and the proof is completed. ∎

(4.5) *Note.* In [16, 17], Miers investigated Lie homomorphisms between operator algebras.

We now characterise those C^*-algebras which have isometrically isomorphic unitary groups.

(4.6) THEOREM. *The unitary groups of the C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isometrically isomorphic if and only if $\mathfrak{A}_i$ is the direct sum of closed ideals I_i, J_i $(i=1,2)$ where I_1 and I_2 are *-isomorphic and J_1 and J_2 are *-anti-isomorphic.*

Proof. Suppose that $\mathfrak{A}_i = I_i \oplus J_i$ $(i=1,2)$ are as in the statement of the theorem. Let $T: I_1 \to I_2$, $S: J_1 \to J_2$ be a *-isomorphism and *-anti-isomorphism, respectively, and let p be the identity of I_1. Then $R: U(\mathfrak{A}_1) \to U(\mathfrak{A}_2)$ is an isometric isomorphism, where

$$R(x) = T(px) + S((1-p)x^*).$$

Conversely, suppose that $Q: U(\mathfrak{A}_1) \to U(\mathfrak{A}_2)$ is an isometric isomorphism. Then the map $\beta: \operatorname{Sk} \mathfrak{A}_1 \to \operatorname{Sk} \mathfrak{A}_2$ of (4.1)(ii) is an isometric Lie isomorphism. Let $\alpha: \operatorname{Her} \mathfrak{A}_1 \to \operatorname{Her} \mathfrak{A}_2$ be given by: $i\alpha(h) = \beta(ih)$. Let $\mathfrak{B}_1$ be a maximal, abelian, C^*-algebra of $\mathfrak{A}_1$. Then $U(\mathfrak{B}_1)$ is a maximal abelian subgroup of $U(\mathfrak{A}_1)$, and so therefore $Q(U(\mathfrak{B}_1))$ is a maximal abelian subgroup of $U(\mathfrak{A}_2)$. It follows that $Q(U(\mathfrak{B}_1)) = U(\mathfrak{B}_2)$, where $\mathfrak{B}_2$ is the (abelian) C^*-subalgebra of $\mathfrak{A}_2$ generated by $Q(U(\mathfrak{B}_1))$. It follows that α, restricted to $\operatorname{Her} \mathfrak{B}_1$, is a real linear isometry onto $\operatorname{Her} \mathfrak{B}_2$, and hence by the Banach–Stone theorem, we can write $\alpha(x) = w\gamma(x)$ $(x \in \operatorname{Her} \mathfrak{B}_1)$, where $\alpha(1) = w$ is a hermitian unitary, and γ is an isomorphism. Since β is a Lie isomorphism, w belongs to the centre of $\mathfrak{A}_2$. Write $w = p_2 - (1-p_2)$; then p_2 is a central projection in $\mathfrak{B}_2$. Let $p_1 \in \operatorname{Her} \mathfrak{B}_1$ be given by: $p_2 = w^{-1}\alpha(p_1) = \gamma(p_1)$. Then p_1 is a central projection in $\mathfrak{A}_1$. Let $I_i = p_i \mathfrak{A}_i$, $J_i = (1-p_i)\mathfrak{A}_i$ $(i=1,2)$. Let $\alpha_I = \alpha|_{\operatorname{Her} I_1}$ and $\alpha_J = \alpha|_{\operatorname{Her} J_1}$. Let $h \in \operatorname{Her} \mathfrak{A}_1$. Then h is in some subalgebra $\mathfrak{B}_1$ as above, and one readily checks that $\alpha_I(h) = p_2\gamma(h) = \gamma(h)$ if $h \in \operatorname{Her} I_1$, and that $\alpha_J(h) = -(1-p_2)\gamma(h) = -\gamma(h)$ if $h \in \operatorname{Her} J_1$. So $\alpha_I(h^2) = (\alpha_I(h))^2$ and α_I is a linear isometry onto $\operatorname{Her} I_2$. Extending α_I in the obvious way to a complex linear map $T: I_1 \to I_2$ and using elementary algebraic manipulations and the fact that β is a Lie homomorphism, we obtain that T is a $*$-isomorphism. Now extend α_J to a linear bijection T' from J_1 onto J_2 and set $S = -T'$. If $h, k \in \operatorname{Her} J_1$, then $S(h^2) = S(h)^2$, and

$$S(i(hk - kh)) = i\beta((ih)(ik) - (ik)(ih)) = -i[S(h)\,S(k) - S(k)\,S(h)].$$

It follows that S is a $*$-anti-isomorphism. ∎

(4.7) Corollary. *The C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $*$-isomorphic if and only if there exists an isometric isomorphism Q from $U(\mathfrak{A}_1)$ onto $U(\mathfrak{A}_2)$ such that $Q(\lambda 1) = \lambda 1$ for all $\lambda \in \mathbb{T}$.*

Proof. If the isomorphism Q is as in the above statement, then, in the notation of the preceding proof, $\alpha(1) = 1 = w$ and so $J_1 = J_2 = \{0\}$. ∎

(4.8) Corollary. *Two von Neumann algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ are Jordan $*$-isomorphic if and only if $U(\mathfrak{A}_1)$ is isometrically isomorphic with $U(\mathfrak{A}_2)$.*

Proof. From [14], every Jordan $*$-isomorphism between von Neumann algebras is a "sum" of a $*$-isomorphism and a $*$-anti-isomorphism. ∎

(4.9) *Note.* In [2], Connes produces a family of mutually non-isomorphic factors $Q_{\lambda,p,\gamma}$, where $\lambda \in (0, 1)$, $p \in \mathbb{P}$ and $\gamma \in \mathbb{C}$ is such that $\gamma^p = 1$. Further, if $\gamma^2 \neq 1$, then $Q_{\lambda,p,\gamma}$ is anti-isomorphic with $Q_{\lambda,p,\bar{\gamma}}$. It follows that there exist non-isomorphic, anti-isomorphic C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$. As noted in [5, Lemma 7], it is a simple corollary of [21, 4.1.21] that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $*$-anti-isomorphic. So there exist *non*-isomorphic C^*-agebras whose unitary groups are isometrically isomorphic. (Using the Dixmier–Douady result, there are much simpler examples of non-isomorphic but $*$-anti-isomorphic C^*-algebras. This seems to be "folklore.")[1]

5. Characterisations of Unitary Groups

Let (G, d) be a metric group and π_d, on the Hilbert space $\mathfrak{H}_d$, be the universal representation $\oplus\{\pi_\phi : \phi \in P_d(G)\}$ of $\operatorname{Rep}_d(G)$. The C^*-subalgebra of $B(\mathfrak{H}_d)$ generated by $\pi_d(G)$ is denoted by $\mathfrak{A}_d$. (We note that if G is abelian, then $\mathfrak{A}_d$ can be identified with $C(\hat{G}_d)$.) We can identify $P_d(G)$ with $S(\mathfrak{A}_d)$. We define $\mathfrak{g}_u$ by

$$\mathfrak{g}_u = \{X \in \operatorname{Sk}(\mathfrak{A}_d) \colon \exp(tX) \in \pi_d(G) \text{ for all } t \in \mathbb{R}\}.$$

[1] The author is indebted to the referee for pointing out that J. Feldman (*Ann. of Math.* **63** (1956), 565–571) proves (4.8) for finite, type II von Neumann algebras without the isometry condition.

Clearly $\mathfrak{g}_u$ is a subset of $\mathfrak{A}_d$; it can be regarded as the "unitary" Lie algebra of G.

(5.1) PROPOSITION. *The set* $\mathfrak{g}_u$ *is a Lie subalgebra of* $\mathrm{Sk}(\mathfrak{A}_d)$.

Proof. Follow the classical Lie group argument. ∎

The space $\mathfrak{g}_u$ is not in general closed in $\mathrm{Sk}(\mathfrak{A}_d)$, but is easily seen to be so if $\pi_d(G)$ is closed in $\mathfrak{A}_d$.

Now define a pseudo-metric d_0 on G by: $d_0(x,y)=\|\pi_d(x)-\pi_d(y)\|$. (In terms of $P_d(G)$, $d_0(x,y)=\sup\{[2(1-\mathrm{Re}\,\phi(y^{-1}x)]^{1/2}: \phi\in P_d(G)\}$.) Clearly $d_0\leqslant d$. We now characterise the identity component of a unitary group.

(5.2) THEOREM. *A connected metric group* (G,d) *is isometrically isomorphic with* $U(\mathfrak{A})_e$ *for some* C^**-algebra* $\mathfrak{A}$ *if and only if*:

(i) d *is complete*;

(ii) $d=d_0$;

(iii) *there exists a closed ideal* J *in* $\mathfrak{A}_d$ *such that* $\mathfrak{A}_d$ *is the vector space direct sum* $J\oplus(\mathfrak{g}_u)_{\mathbb{C}}$, *and* $\mathrm{dist}(J,1-u)=\|1-u\|$ *for all* $u\in\pi_d(G)$.

Proof. Suppose that (i)–(iii) hold. Then (G,d) can be identified with the closed, connected subgroup $\pi_d(G)$ of $U(\mathfrak{A}_d)$. Let $\mathfrak{A}=\mathfrak{A}_d/J$ and $\pi\colon\mathfrak{A}_d\to\mathfrak{A}$ be the quotient map. From (iii), we see that $\pi|_G$ is an isometric isomorphism from G to a closed connected subgroup H of $U(\mathfrak{A})_e$, and that $\pi(\mathfrak{g}_u)=\mathrm{Sk}\,\mathfrak{A}$. So H contains a neighbourhood of 1 in $U(\mathfrak{A})_e$ and hence $H=U(\mathfrak{A})_e$.

Conversely, suppose that $G=U(\mathfrak{A})_e$ for some C^*-algebra $\mathfrak{A}$. Trivially, (i) and (ii) hold. Represent $\mathfrak{A}$ faithfully on a Hilbert space $\mathfrak{H}$. We can write $\mathfrak{H}=\oplus\{\mathfrak{H}_c: c\in C\}$, where $\mathfrak{A}|_{\mathfrak{H}_c}$ has a cyclic vector. So the representtion π_c of G, where $\pi_c(x)=x|_{\mathfrak{H}_c}$, is cyclic, and so is equivalent to some π_{ϕ_c} $(\phi_c\in P_d(G))$. Considering $\oplus\{\pi_{\phi_c}: c\in C\}$, we see that there is a $*$-homomorphism P from $\mathfrak{A}_d$ onto $\mathfrak{A}$ such that $P|_{\pi_d(G)}$ is an isometry onto G. Let $J=\ker P$. Then $\mathrm{dist}(J,1-u)=\|1-u\|$ for all $u\in\pi_d(G)$. Further, if $x\in\mathfrak{g}_u$ and $t\in\mathbb{R}$, then

$$\|1-\exp(tx)\|=\|1-\exp(tP(x))\|$$

and it follows that P, restricted to $\mathfrak{g}_u$, is an isometry into $\mathrm{Sk}\,\mathfrak{A}$. That $P(\mathfrak{g}_u)=\mathrm{Sk}\,\mathfrak{A}$ follows by noting that there is a natural correspondence between the sets of one-parameter groups in G and $\pi_d(G)$. It now follows that $\mathfrak{A}_d=J\oplus(\mathfrak{g}_u)_{\mathbb{C}}$ and (iii) is established. ∎

We now determine $\mathfrak{g}_u$ when (G,d) is a connected Lie group with d left invariant. Let $H = \ker \pi_d$ and ρ be the (left invariant) metric on G/H given by:

$$\rho(xH, yH) = \inf\{d(a, b): a \in xH, b \in yH\}.$$

The metric ρ gives the quotient topology of G/H.

(5.3) Proposition. *The Lie group G/H is of the form $\mathbb{R}^n \times K$, where K is a compact group. Further, $\mathfrak{g}_u$ is isomorphic with the Lie algebra of G/H (so that in particular $\mathfrak{g}_u$ is finite-dimensional).*

Proof. The mapping $\pi \to \alpha(\pi)$, where $\alpha(\pi)(xH) = \pi(x)$, defines a bijection from $\mathrm{Rep}_d(G)$ onto $\mathrm{Rep}_\rho(G/H)$. Now $\mathrm{Rep}_\rho(G/H) \subset \mathrm{Rep}_n(G/H)$, and $\mathrm{Rep}_\rho(G/H)$ separates the points of G/H. So by [6, 17.4.4; 22], $G/H = \mathbb{R}^n \times K$ for some compact group K. We can suppose for the rest of the argument that $H = \{e\}$. We define a Lie homomorphism Q: $\mathfrak{g} \to \mathfrak{g}_u$, where $\mathfrak{g}$ is the Lie algebra of G. If $X \in \mathfrak{g}$, then the map $t \to \pi_d(\exp(tX))$ is a norm continuous, one-parameter group in $U(\mathfrak{A}_d)$, so that we find a unique element $Q(X) \in \mathrm{Sk}\, \mathfrak{A}_d$ with $\pi_d(\exp(tX)) = \exp(tQ(x))$. It is routine to check that Q is a Lie homomorphism from $\mathfrak{g}$ into $\mathfrak{g}_u$. Since π_d is faithful, Q is injective. It remains to show that $Q(\mathfrak{g}) = \mathfrak{g}_u$. Let $Z \in \mathfrak{g}_u$ and define β: $\mathbb{R} \to G$ by: $\beta(t) = \pi_d^{-1}(\exp(tZ))$. If C is a compact subset of G, then the topologies on C induced by d and d_0 coincide. Since β is d_0-continuous and G is σ-compact, it follows that $\beta^{-1}(D)$ is measurable if D is closed in (G, d). So β is a measurable homomorphism from $\mathbb{R}$ into (G, d), and so is d-continuous [10, (22, 18)]. So $Z = Q(X)$, where $\exp(tX) = \beta(t)$. ∎

The remainder of this paper is devoted to characterisations of *abelian* unitary groups. The first of these involves an action by $\mathbb{T}$-valued functions on the group.

Let $H_n = U(C(\mathbb{T}^n))$ $(n \geqslant 1)$. An *action* of $\{H_n\}$ on $\{G^n\}$, where (G, d) is an abelian metric group, is a sequence of functions $\{\Phi_n\}$, where Φ_n maps H_n into the set of maps from G^n to G. If $f \in H_n$ and $u_1, \ldots, u_n \in G$, then we shall write $f(u_1, \ldots, u_n)$ instead of $\Phi_n(f)(u_1, \ldots, u_n)$. Now define

$$X = \{\gamma \in \hat{G}_d : \gamma(f(u_1, \ldots, u_n)) = f(\gamma(u_1), \ldots, \gamma(u_n)) \quad \text{for all } n \geqslant 1, f \in H_n \text{ and } u_1, \ldots, u_n \in G\}. \tag{5}$$

For non-empty $Y \subset \hat{G}_d$, define a pseudo-metric d_Y on G by $d_Y(x, y) = \sup\{|\gamma(x) - \gamma(y)|: \gamma \in Y\}$. Obviously, d_Y is invariant and $d_Y \leqslant d$.

For the moment we are concerned with the case in which $d_X = d$. This equality implies a number of pleasant properties of the action $\{\Phi_n\}$: for example, if $f \in H_n$ is of the form $f(z_1, \ldots, z_n) = z_1^{N_1} \cdots z_n^{N_n}$, then $f(u_1, \ldots, u_n) = u_1^{N_1} \cdots u_n^{N_n}$ for $u_1, \ldots, u_n \in G$.

(5.4) THEOREM. *Let (G, d) be a complete, connected, abelian metric group. Then G is the identity component of the unitary group of a C^*-algebra if and only if there exists an action $\{\Phi_n\}$ of $\{H_n\}$ on $\{G^n\}$ such that $d_X = d$.*

Proof. Suppose that $G = U(C(W))_e$, where W is a compact Hausdorff space. If $f \in H_n$ and $u_1, ..., u_n \in G$, then we define $\Phi_n(f)$ by

$$\Phi_n(f)(u_1, ..., u_n)(w) = f(u_1(w), ..., u_n(w)) \qquad (w \in W).$$

The uniform continuity of f entails that $\Phi_n : G^n \to U(C(W))$ is continuous. So $\Phi_n(G^n)$ is connected, and contains $f(1, 1, ..., 1)\ 1$. So $\Phi_n(G^n) \subset G$, and $\{\Phi_n\}$ is an action of $\{H_n\}$ on $\{G^n\}$. Now, in the notation of (5), every point of W can be identified with a point of X, and it follows that $d_X = d$.

Conversely, suppose that $\{\Phi_n\}$ is an action of $\{H_n\}$ on $\{G^n\}$ such that $d_X = d$. By considering the map $u \to \hat{u}|_X$ from G into $C(X)$ and using the completeness of G, we can identify G with a closed subgroup of $U(C(X))_e$. It suffices to show that $G = U(C(X))_e$. Let $\varepsilon \in (0, \pi/2)$ and $h \in \mathrm{Her}(C(X))$ be such that $\mathrm{Sp}(h) \subset [-\pi + 2\varepsilon, \pi - 2\varepsilon]$. By the Stone–Weierstrass theorem, we can find a sequence $\{w_m\}$ with

$$w_m \to \exp(ih), \quad \text{where} \quad w_m = \sum_1^{N_m} \lambda_i^{(m)} u_i^{(m)} \quad \text{with} \quad \lambda_i^{(m)} \in \mathbb{C} \quad \text{and} \quad u_i^{(m)} \in G.$$

We can suppose that $0 \notin \mathrm{Sp}(w_m)$ for all m and that

$$\mathrm{Sp}(w_m/|w_m|) \subset \{\exp(i\theta): -\pi + \varepsilon \leqslant \theta \leqslant \pi - \varepsilon\} = A.$$

Let $W_m = \{(u_1^{(m)}(x), ..., u_{N_m}^{(m)}(x)): x \in X\}$ and $F_m : W_m \to A$ be given by $F_m(z_1, ..., z_{N_m}) = (\sum \lambda_i^{(m)} z_i)/|\sum \lambda_i^{(m)} z_i|$. Identifying A with $[-\pi + \varepsilon, \pi - \varepsilon]$ and using the Tietze extension theorem, we can find $g_m \in H_{N_m}$ extending F_m. Let $u_m = g(u_1, ..., u_{N_m})$. Then for $\gamma \in X$, we have $\hat{u}_m(\gamma) = (w_m/|w_m|)(\gamma)$, so that $u_m = w_m/|w_m|$. Since $w_m \to \exp(ih)$, it follows that $u_m \to \exp(ih)$. So $\exp(ih) \in G$. It follows that $G = U(C(X))_e$. ∎

The next result gives a straightforward "harmonic analysis" characterisation of unitary abelian groups. Let (G, d) be a metric abelian group. We use the notations d_0 and d_Y of (5.1) and (5.3). A subset Y of $\hat{G}_d$ is called a *boundary* (for G) if $d_Y = d_0$. (With the notion of "boundary" for function spaces in mind, we can say that Y is a boundary for the set of functions of the form $(\hat{x} - \hat{y})|_{\hat{G}_d}$ $(x, y \in G)$.)

In general, there are many boundaries for G. Even in the very simple case $G = \mathbb{T}$ with the usual metric, we have $\{1\}$ and $\{-1\}$ as the minimal boundaries.

A subset Γ of $\hat{G}_d$ is called a *Kronecker set* if whenever $u: \Gamma \to \mathbb{T}$ is continuous and $\varepsilon > 0$, then there exists $x \in G$ with

$$\sup_{\gamma \in \Gamma} |\hat{x}(\gamma) - u(\gamma)| < \varepsilon.$$

(See [20, pp. 97–98].)

(5.5) THEOREM. *Conditions* (i) *and* (ii) *are equivalent for a complete, metric, abelian group* (G, d):

(i) *(G, d) is the unitary group of a C^*-algebra*;

(ii) *$d = d_0$ and there exists a boundary, Kronecker set for G.*

Proof. Suppose that (i) holds, and that $G = U(C(X))$ for some compact Hausdorff space X. Then X can be identified with a boundary, Kronecker set of G.

Conversely, suppose that (ii) holds, and let X be a boundary, Kronecker set for G. We can suppose that X is closed in $\hat{G}_d$ and so is compact ((3.3)(ii)). The map $u \to \hat{u}|_X$ is an isometric isomorphism onto $U(C(X))$. ∎

It follows from (5.5) and (4.6) that any two boundary Kronecker sets for a unitary abelian group are homeomorphic.

Condition (ii) is obviously very strong. A deeper version of (5.5) is proved in (5.6). We require two preliminary results. The first of these is rather curious and is of intrinsic interest. Recall that $H^0(X, \mathbb{Z}_2)$ can be identified with the multiplicative group of continuous functions from X into $\{-1, 1\}$.

(5.6) PROPOSITION [13, Chap. 4]. *Let X be a compact Hausdorff space, and let*

$$B = \{a \in C(X)^{-1}: \|a\| < 2\},$$
$$\tilde{B} = \{(u, v) \in U(C(X)) \times U(C(X)): (u - v) \in B\}.$$

Let $p: \hat{B} \to B$ be given by: $p((u, v)) = u - v$. Then $(\tilde{B}, p, B)$ is a trivial, principal $H^1(X, Z_2)$-bundle.

Proof. We have to show that $(\tilde{B}, p, B)$ is bundle-isomorphic to the product principle bundle $(B \times H^0(X, \mathbb{Z}_2), q, B)$.

We first define two maps H, L from $D = \{z \in \mathbb{C}: 0 < |z| < 2\}$ to $\mathbb{T} \times \mathbb{T}$ as follows. If $z \in D$, there are exactly two chords of the circle $|z| = 1$ which are parallel and equal in length to $0z$. Analytically, the extremities of these chords are a_z, b_z and c_z, d_z, where

$$\begin{aligned}(a_z, b_z) &= ([|z|^2 + i|z|(4-|z|^2)^{1/2}]/2\bar{z},\\ &\qquad [-|z|^2 + i|z|(4-|z|^2)^{1/2}]/2\bar{z})\\ \text{and} \qquad (c_z, d_z) &= ([|z|^2 - i|z|(4-|z|^2)^{1/2}]/2\bar{z},\\ &\qquad [-|z|^2 - i|z|(4-|z|^2)^{1/2}]/2\bar{z}).\end{aligned} \tag{6}$$

Define functions F_1, F_2, G_1, G_2 from D into $\mathbb{T}$ by $F_1(z) = a_z$, $F_2(z) = b_z$, $G_1(z) = c_z$, and $G_2(z) = d_z$, and define F, G from D into $\mathbb{T} \times \mathbb{T}$ by $F(z) = (F_1(z), F_2(z))$ and $G(z) = (G_1(z), G_2(z))$.

Define $\alpha: B \times H^0(X, \mathbb{Z}_2) \to \tilde{B}$ by

$$\begin{aligned}\alpha((a,f))(x) &= F(a(x)) \qquad \text{if} \quad f(x) = 1,\\ &= G(a(x)) \qquad \text{if} \quad f(x) = -1.\end{aligned}$$

Now if $\alpha((a,f)) = (u, v)$, then, for each $x \in X$, $u(x) - v(x)$ is either $F_2(x) - F_1(x)$ or $G_2(x) - G_1(x)$, and so is $a(x)$. So $u - v = a$, and $p \circ \alpha = q$. It remains to show that α is a homeomorphism onto $\tilde{B}$.

Let $(u, v) \in \tilde{B}$ and $a = u - v$. Then for each $x \in X$, $(u(x), v(x)) \in \{F(a(x)), G(a(x))\}$. Since there exists $\varepsilon > 0$ such that $0 < \varepsilon < |a(x)| < 2 - \varepsilon$, it follows that there exists $\eta > 0$ such that $|F_i(a(x)) - G_i(a(x))| > \eta$ for all $x \in X$ and $i = 1, 2$. So if $A = \{x: (u(x), v(x)) = F(a(x))\}$ and $C = \{x: (u(x), v(x)) = G(a(x))\}$, then $A \mid C$ partitions X. Setting $f = \chi_A - \chi_C$, we obtain that $\alpha((a,f)) = (u, v)$. So α is surjective. It is trivial that α is continuous and one-to-one. It remains to show that α^{-1} is continuous. Suppose that $\alpha((a_n, f_n)) \to \alpha((a,f))$ in $\tilde{B}$. Then $a_n = p(\alpha((a_n, f_n))) \to p(\alpha((a,f))) = a$. We have to show that $f_n = f$ eventually. Write $f_n = p_n - (1 - p_n)$, where p_n is a projection in $C(X)$. Then $\alpha((a_n, f_n)) = (F_1(a_n) p_n + G_1(a_n)(1 - p_n),\ F_2(a_n) p_n + G_2(a_n)(1 - p_n))$. We can suppose that for some $\varepsilon > 0$, $0 < \varepsilon \| a_n \| < 2 - \varepsilon$: then $\eta = \inf\{|F_1(t) - G_1(z)|: 0 < \varepsilon \leqslant |z|, |t| \leqslant 2 - \varepsilon\} > 0$. If $(1 - p_n) p(x_n) = 1$ for n in an infinite set K and some $x_n \in X$, then

$$\eta \leqslant |([F_1(a_n) p_n + G_1(a_n)(1 - p_n)] - [F_1(a) p + G_1(a)(1 - p)])(x_n)|$$

and a contradiction of the fact that $\alpha((a_n, f_n)) \to \alpha((a,f))$ results. So $(1 - p_n) p = 0$ eventually. Similarly, $p_n(1 - p) = 0$ eventually, and so $p_n = p$ eventually. This completes the proof. ∎

(5.7) *Note.* It is of interest to enquire if there is an analogue of (5.6) for a general C^*-algebra $\mathfrak{A}$. We can define B, $\tilde{B}$, and p as in (5.6), with $C(X)$ replaced by $\mathfrak{A}$. If $a \in B$, then we have the polar decomposition $a = u'h$, where $u' \in U(\mathfrak{A})$ and h is an invertible positive element with $\|h\| < 2$. (Here $h = |a|$ and $u' = a|a|^{-1}$.) Using (5.6), we can find u_1, v_1 in a commutative $C^* -$ subalgebra of $\mathfrak{A}$ containing h such that $h = u_1 - v_1$. So

$a = u'u_1 - u'v_1$, and it follows that $p(\tilde{B}) = B$. Under what circumstances are all representations $(u - v)$ of a of this form? This is the case if $\mathfrak{A} = M_2$, the algebra of 2×2 complex matrices. (To see this, let $h = u - v$, where $h \in \mathfrak{A}^{-1}$, $0 < h < 2$ and $u, v \in U(\mathfrak{A})$. Write $u = u_1 + iw$, $v = v_1 + iw$ with $u_1, v_1, w \in \operatorname{Her} \mathfrak{A}$. Since w commutes with both u_1 and v_1, we have to show that $u_1 v_1 = v_1 u_1$. Since u and v are unitary, we have $u_1^2 = 1 - w^2 = v_1^2$. We can suppose that

$$1 - w^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

If $\lambda_1 \neq \lambda_2$, then every square root of $(1 - w^2)$ is diagonal, and so $u_1 v_1 = v_1 u_1$. If $\lambda_1 = \lambda_2$, then $1 - w^2 = \lambda_1 I$. If one of u_1, v_1 is $\pm \sqrt{\lambda_1}\, I$, then u_1, v_1 commute. Otherwise $\operatorname{trace} u_1 = \operatorname{trace} v_1 = 0$, and it follows that $\operatorname{trace} h = 0$, implying that h has a negative eigenvalue. This is a contradiction.)

(5.8) PROPOSITION. *Let X be a compact metric space, $\mathfrak{A} = C(X)$ and $\operatorname{Sing} \mathfrak{A}$ be the set of non-invertible elements of $\mathfrak{A}$. Then $(U(\mathfrak{A}) - U(\mathfrak{A})) \cap \operatorname{Sing} \mathfrak{A}$ has empty interior in $\mathfrak{A}$.*

Proof. Suppose that $(U(\mathfrak{A}) - U(\mathfrak{A}) \cap \operatorname{Sing} \mathfrak{A}$ has interior in $\mathfrak{A}$. Then we can find $g \in \mathfrak{A}$ and $\varepsilon > 0$ such that the set $A = \{F \in \mathfrak{A}: \|F - g\| < \varepsilon\} \subset (U(\mathfrak{A}) - U(\mathfrak{A})) \cap \operatorname{Sing} \mathfrak{A}$. Let $W = \{x \in X: g(x) \neq 0\}$. We shall show that W is closed in X. Suppose not. Then we can find $x_0 \in X$ with $g(x_0) = 0$ and such that $V \cap W \neq \varnothing$ for every neighbourhood V of x_0. It follows that there is a sequence $\{x_n\}$ in X with $x_n \to x_0$, and for all n, $0 < |g(x_{n+1})| < |g(x_n)|$. Now $g = u - v$, where $u, v \in U(\mathfrak{A})$. By (5.6), if $x \in W$, then $\{(u(x), v(x))\} \in \{(a_z, b_z), (c_z, d_z)\}$, where $z = g(x)$. Consideration of Eqs. (6), together with the existence of $\lim u(x_n)$, yields that $\lim(g(x_n)/|g(x_n)|)$ exists. Find a continuous, real-valued function f on $\mathbb{R}$ such that $\{\exp(if(|g(x_n)|^{-1}))\}$ does not converge. (Note that $|g(x_n)|^{-1} \uparrow \infty$). Now find relatively compact open neighbourhoods V_1, V_2 of x_0 such that $V_1^- \subset V_2$, and $|g(x)| < \frac{1}{2}\varepsilon$ when $x \in V_2$. Let $r \in C(X)$ be such that $0 \leqslant r \leqslant 1$ and $r(x) = 1$ if $x \in V_1$ and $r(x) = 0$ if $x \notin V_2$. Define

$$h(x) = g(x) \exp(i[r(x) f(|g(x)|^{-1})]) \qquad (x \in X).$$

(Of course, when $g(x) = 0$, we take $h(x) = 0$.) It is obvious that $h \in C(X)$ and that $\|h - g\| < \varepsilon$. So $h \in A$ and $h(x_0) = 0$. Hence $\lim h(x_n)/|h(x_n)|$ exists, and therefore so does

$$\begin{aligned} &\lim (\exp i[f(|g(x_n)|^{-1})]) \\ &\qquad = \lim [(h(x_n)/|h(x_n)|)/(g(x_n)/|g(x_n)|)]. \end{aligned}$$

This is a contradiction.

It follows that W is open and closed in X. But then the function k, where $k(x)=g(x)$ if $x \in W$, and $k(x)=\frac{1}{2}\varepsilon$ if $x \notin W$, is an invertible function in A, giving a contradiction. ∎

(5.9) *Note.* Note that there are simple spaces X such that Sing $C(X)$ has interior [8].

(5.10) THEOREM. *Let* (G, d) *be a complete, connected, separable, abelian, metric group. Then* (i) *and* (ii) *are equivalent*:

(i) $G=U(C(X))_e$, *where* X *is compact metric.*

(ii) $d=d_0$, *and there exists a closed boundary* Γ *for* G *such that the set* $\{(\hat{x}-\hat{y})|_\Gamma : x, y \in G\}$ *has a non-empty interior in* $C(\Gamma)$.

Proof. Suppose that (i) holds. As usual, we take $\Gamma=X$ and have $d=d_0$. In the notation of (5.6), the set $(U(C(X))_e \times U(C(X))_e) \cap \tilde{B}$ is open and non-empty in $\tilde{B}$. It follows from (5.6) that the last condition of (ii) is satisfied.

Conversely, suppose that (ii) holds. Let $X=\Gamma$, and as usual, identify G with a closed, connected, separable subgroup of $U(C(X))$. Since G is separable, so is $(G-G)$, and hence $C(X)$ contains a non-void, open, separable set. Hence $C(X)$ is separable, and X is compact metric [7, p. 437, Exercise 17]. It remains to show that $G=U(C(X))_e$. From (5.8), we have that $(G-G) \cap B$ has non-empty interior in $C(X)$. Find open balls U, V in $C(X)$ such that

$$U \subset U^- \subset V \subset (G-G) \cap B.$$

Now $H^0(X, \mathbb{Z}_2)$ is countable; indeed every open and closed subset of X is a finite union of members of a given countable base for X. So by (5.6), there is a countable family $\mathscr{F}$ of disjoint open subsets of $\tilde{B}$ with $p^{-1}(V)=\bigcup\mathscr{F}$, and $p|_W$ a homeomorphism from W onto V for all $W \in \mathscr{F}$. Let

$$\mathscr{G}=\{U^- \cap p(W \cap (G \times G)) \colon W \in \mathscr{F}\}.$$

Then $\mathscr{G}$ is a countable family of closed subsets of U^-, and $U^-=\bigcup\mathscr{G}$. We can find, by the Baire category theorem, an element $Z \in \mathscr{G}$ with non-empty interior in B. So $(G \times G) \cap \tilde{B}$ has interior in $\tilde{B}$, and since $\tilde{B}$ is open in $U(C(X)) \times U(C(X))$, we see that $G \times G$ has interior in $U(C(X)) \times U(C(X))$. Hence G has interior in $U(C(X))$, and since $G \subset U(C(X))_e$ and $U(C(X))$ is connected, we have $G=U(C(X))_e$. ∎

We conclude by listing some problems which seem to be open.

(1) Can one determine $\hat{G}_d$ when G is the unitary group of a C^*-algebra? (The abelian and von Neumann algebra cases are dealt with in (4.2) and (4.4). See also (4.8).)

(2) What can one say about C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which are such that there is a (topological and algebraic) isomorphism from $U(\mathfrak{A}_1)$ onto $U(\mathfrak{A}_2)$? (See (4.6) for the isometric case.)

(3) Is (G, d) a unitary group if every maximal, abelian subgroup is a unitary group? (See [3] for a solution to the analogous question for Banach *-algebras.)

(4) Can one find reasonable properties of an action of $\{H_n\}$ on $\{G^n\}$ which will ensure that $d_X = d$? (See (5.3), ff.)

(5) Is there an analogue of (5.6) for C^*-algebras in general? (See (5.7).)

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